

INTERSECTING FREE SUBGROUPS IN FREE PRODUCTS OF LEFT ORDERED GROUPS

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ABSTRACT. A conjecture of Dicks and the author on rank of the intersection of factor-free subgroups in free products of groups is proved for the case of left ordered groups.

1. INTRODUCTION

Recall that the Hanna Neumann conjecture [16] claims that if F is a free group of rank $r(F)$, $\bar{r}(F) := \max(r(F) - 1, 0)$ is the reduced rank of F , and H_1, H_2 are finitely generated subgroups of F , then $\bar{r}(H_1 \cap H_2) \leq \bar{r}(H_1)\bar{r}(H_2)$. It was shown by Hanna Neumann [16] that $\bar{r}(H_1 \cap H_2) \leq 2\bar{r}(H_1)\bar{r}(H_2)$. For more discussion and results on this problem, the reader is referred to [2], [3], [6], [15], [17], [18].

More generally, let $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ be the free product of some groups G_α , $\alpha \in I$. According to the classic Kurosh subgroup theorem [13], every subgroup H of \mathcal{F} is a free product $F(H) * \prod_{\alpha \in I}^* s_{\alpha, \beta} H_{\alpha, \beta} s_{\alpha, \beta}^{-1}$, where $H_{\alpha, \beta}$ is a subgroup of G_α , $s_{\alpha, \beta} \in \mathcal{F}$, and $F(H)$ is a free subgroup of \mathcal{F} such that, for every $s \in \mathcal{F}$ and $\gamma \in I$, it is true that $F(H) \cap sG_\gamma s^{-1} = \{1\}$. We say that H is a *factor-free* subgroup of \mathcal{F} if $H = F(H)$ in the above form of H , i.e., for every $s \in \mathcal{F}$ and $\gamma \in I$, we have $H \cap sG_\gamma s^{-1} = \{1\}$. Since a factor-free subgroup H of \mathcal{F} is free, the reduced rank $\bar{r}(H) := \max(r(H) - 1, 0)$ of H is well defined. Let $q^* = q^*(G_\alpha, \alpha \in I)$ denote the minimum of orders > 2 of finite subgroups of groups G_α , $\alpha \in I$, and $q^* := \infty$ if there are no such subgroups. If $q^* = \infty$, define $\frac{q^*}{q^*-2} := 1$. It was shown by Dicks and the author [4] that if H_1, H_2 are finitely generated factor-free subgroups of \mathcal{F} , then

$$\bar{r}(H_1 \cap H_2) \leq 2 \frac{q^*}{q^*-2} \bar{r}(H_1) \bar{r}(H_2).$$

It was conjectured by Dicks and the author [4] that if groups G_α , $\alpha \in I$, contain no involutions then, similarly to the Hanna Neumann conjecture, the coefficient 2 could be left out and

$$\bar{r}(H_1 \cap H_2) \leq \frac{q^*}{q^*-2} \bar{r}(H_1) \bar{r}(H_2). \quad (1)$$

A special case of this generalization of the Hanna Neumann conjecture is established by Dicks and the author [5] by proving that if H_1, H_2 are finitely generated factor-free subgroups of the free product \mathcal{F} all of whose factors are groups of order 3, then, indeed, $\bar{r}(H_1 \cap H_2) \leq \frac{q^*}{q^*-2} \bar{r}(H_1) \bar{r}(H_2) = 3\bar{r}(H_1) \bar{r}(H_2)$. We remark that it follows from results of [4] that the last inequality (as well as (1)) is sharp and may not be improved. Here is another special case when the conjectured inequality (1) holds true.

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Theorem 1. *Suppose that G_α , $\alpha \in I$, are left (or right) ordered groups, $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ is their free product, and H_1, H_2 are finitely generated factor-free subgroups of \mathcal{F} . Then*

$$\bar{r}(H_1 \cap H_2) \leq \frac{q^*}{q^* - 2} \bar{r}(H_1) \bar{r}(H_2) = \bar{r}(H_1) \bar{r}(H_2).$$

Moreover, let $S(H_1, H_2)$ denote a set of representatives of those double cosets $H_1 t H_2$ of \mathcal{F} , $t \in \mathcal{F}$, that have the property $H_1 \cap t H_2 t^{-1} \neq \{1\}$. Then

$$\bar{r}(H_1, H_2) := \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap s H_2 s^{-1}) \leq \bar{r}(H_1) \bar{r}(H_2).$$

We remark that Antolín, Martino, and Schwabrow [1] proved a more general result on Kurosh rank of the intersection of subgroups of free products of right ordered groups by utilizing the Bass–Serre theory of groups acting on trees and some ideas of Dicks [3]. Our proof of Theorem 1 seems to be of independent interest as it uses explicit geometric construction of graphs, representing subgroups of free products, that are often more suitable for counting arguments, see [9], [11].

It is fairly easy to see that Theorem 1 implies both the Hanna Neumann conjecture and the strengthened Hanna Neumann conjecture, put forward by W. Neumann [17], see Sect. 5. The strengthened Hanna Neumann conjecture claims that if H_1, H_2 are finitely generated subgroups of a free group F , then

$$\sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap s H_2 s^{-1}) \leq \bar{r}(H_1) \bar{r}(H_2),$$

where the set $S(H_1, H_2)$ is defined as in Theorem 1 with F in place in \mathcal{F} . Recall that Friedman [6] proved the strengthened Hanna Neumann conjecture by making use of sheaves on graphs and Mineyev [15] gave a proof to the strengthened Hanna Neumann conjecture by using Hilbert modules and group actions, see also Dicks’s proof [3]. Similarly to Dicks [3] and Mineyev [15], we also use the idea of group ordering and special sets of edges, however, our arguments deal directly with core graphs of subgroups of free products that are analogous to Stallings graphs [18] representing subgroups of free groups.

In Sect. 2, we introduce necessary definitions and basic terminology. In Sect. 3, we define and study strongly positive words in free products of left ordered groups. Sect. 4 contains the proof of Theorem 1. In Sect. 5, we briefly look at the case of free groups.

2. PRELIMINARIES

Let G_α , $\alpha \in I$, be nontrivial groups, $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ be their free product, and H a finitely generated factor-free subgroup of \mathcal{F} , $H \neq \{1\}$. Consider an alphabet $\mathcal{A} = \cup_{\alpha \in I} G_\alpha$, where $G_\alpha \cap G_{\alpha'} = \{1\}$ if $\alpha \neq \alpha'$.

Similarly to the graph-theoretic approach of article [9], in a simplified version suitable for finitely generated factor-free subgroups of \mathcal{F} , see [7], [8], [10], we first define a labeled \mathcal{A} -graph $\Psi(H)$ which geometrically represents H in a fashion analogous to the way the Stallings graph represents a subgroup of a free group, see [18].

If Γ is a graph, $V\Gamma$ denotes the vertex set of Γ and $E\Gamma$ denotes the set of oriented edges of Γ . If $e \in E\Gamma$, e_- , e_+ denote the initial, terminal, resp., vertices of e and e^{-1} is the edge with opposite orientation, where $e^{-1} \neq e$ for every $e \in E\Gamma$, $(e^{-1})_- = e_+$, $(e^{-1})_+ = e_-$. A finite *path* $p = e_1 \dots e_k$ in Γ is a sequence of edges e_i such that

$(e_i)_+ = (e_{i+1})_-$, $i = 1, \dots, k-1$. Denote $p_- := (e_1)_-$, $p_+ := (e_k)_+$, and $|p| := k$, where $|p|$ is the *length* of p . We allow the possibility $|p| = 0$ and $p = \{p_-\} = \{p_+\}$. A finite path p is called *closed* if $p_- = p_+$. An *infinite* path $p = e_1 \dots e_k \dots$ is an infinite sequence of edges e_i such that $(e_i)_+ = (e_{i+1})_-$ for all $i = 1, 2, \dots$. If $p = e_1 \dots e_k \dots$ and $q = f_1 \dots f_\ell \dots$ are infinite paths such that $(e_1)_- = (f_1)_-$, then $q^{-1}p := \dots f_\ell^{-1} \dots f_1^{-1} e_1 \dots e_k \dots$ is a *biinfinite* path. A path p is *reduced* if p contains no subpath of the form ee^{-1} , $e \in E\Gamma$. A closed path $p = e_1 \dots e_k$ is *cyclically reduced* if $|p| > 0$ and both p and the cyclic permutation $e_2 \dots e_k e_1$ of p are reduced paths. The *core* of a graph Γ , denoted $\text{core}(\Gamma)$, is the minimal subgraph of Γ that contains every edge e which can be included into a cyclically reduced path in Γ .

Let Ψ be a graph whose vertex set $V\Psi$ consists of two disjoint parts $V_P\Psi, V_S\Psi$, so $V\Psi = V_P\Psi \cup V_S\Psi$. Vertices in $V_P\Psi$ are called *primary* and vertices in $V_S\Psi$ are termed *secondary*. Every edge $e \in E\Psi$ connects primary and secondary vertices, hence Ψ is a bipartite graph. Ψ is called a *labeled \mathcal{A} -graph*, or simply *\mathcal{A} -graph* if Ψ is equipped with a map $\varphi : E\Psi \rightarrow \mathcal{A}$, called *labeling*, such that, for every $e \in E\Psi$, $\varphi(e) \in \mathcal{A} = \cup_{\alpha \in I} G_\alpha$, $\varphi(e^{-1}) = \varphi(e)^{-1}$ and if $e_+ = f_+ \in V_S\Psi$, then $\varphi(e), \varphi(f) \in G_\alpha$ for the same $\alpha = \theta(e_+) \in I$, called the *type* of the vertex $e_+ \in V_S\Psi$ and denoted $\alpha = \theta(e_+)$. If $e_+ \in V_S\Psi$, define $\theta(e) := \theta(e_+)$ and $\theta(e^{-1}) := \theta(e_+)$. Thus, for every $e \in \Psi$, we have defined an element $\varphi(e) \in \mathcal{A}$, called the *label* of e , and $\theta(e) \in I$, the *type* of e .

The reader familiar with van Kampen diagrams over a free product of groups, see [14], will recognize that our labeling function $\varphi : E\Psi \rightarrow \mathcal{A}$ is defined in the way analogous to labeling functions on van Kampen diagrams over free products of groups. Recall that van Kampen diagrams are planar 2-complexes whereas graphs are 1-complexes, however, apart from this, the ideas of cancelations and edge foldings work equally well for both diagrams and graphs.

An \mathcal{A} -graph Ψ is called *irreducible* if properties (P1)–(P3) hold true:

- (P1) If $e, f \in E\Psi$, $e_- = f_- \in V_P\Psi$, and $e_+ \neq f_+$, then $\theta(e) \neq \theta(f)$.
- (P2) If $e, f \in E\Psi$, $e \neq f$, and $e_+ = f_+ \in V_S\Psi$, then $\varphi(e) \neq \varphi(f)$ in $G_{\theta(e)}$.
- (P3) Ψ has no multiple edges, $\deg_\Psi v > 0$ for every $v \in V\Psi$, and there is at most one vertex of degree 1 in Ψ which, if exists, is primary.

Suppose Ψ is a connected irreducible \mathcal{A} -graph and a primary vertex $o \in V_P\Psi$ is distinguished so that $\deg_\Psi o = 1$ if Ψ happens to have a vertex of degree 1. Then o is called the *base* vertex of $\Psi = \Psi_o$.

As usual, elements of the free product $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ are regarded as words over the alphabet $\mathcal{A} = \cup_{\alpha \in I} G_\alpha$. A *syllable* of a word W over \mathcal{A} is a maximal subword of W all of whose letters belong to the same factor G_α . The *syllable length* $\|W\|$ of W is the number of syllables of W , whereas the *length* $|W|$ of W is the number of all letters in W . A nonempty word W over \mathcal{A} is called *reduced* if every syllable of W consists of a single letter. Clearly, $|W| = \|W\|$ if W is reduced. An arbitrary nontrivial element of the free product \mathcal{F} can be uniquely written as a reduced word. A word W is called *cyclically reduced* if W^2 is reduced. We write $U = W$ if words U, W are equal as elements of \mathcal{F} . The literal (or letter-by-letter) equality of words U, W is denoted $U \equiv W$.

The significance of irreducible \mathcal{A} -graphs for geometric interpretation of factor-free subgroups H of \mathcal{F} is given in the following.

Lemma 1. *Suppose H is a finitely generated factor-free subgroup of the free product $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$, $H \neq \{1\}$. Then there exists a finite connected irreducible \mathcal{A} -graph $\Psi = \Psi_o(H)$, with the base vertex o , such that a reduced word W over the alphabet \mathcal{A} belongs to H if and only if there is a reduced path p in $\Psi_o(H)$ such that $p_- = p_+ = o$, $\varphi(p) = W$ in \mathcal{F} , and $|p| = 2|W|$.*

Proof. The proof is based on Stallings's folding techniques and is somewhat analogous to the proof of van Kampen lemma for diagrams over free products of groups, see [14] (in fact, it is simpler because foldings need not preserve the property of being planar for the diagram). A more general approach, suitable for an arbitrary subgroup of \mathcal{F} , is discussed in Lemmas 1, 4 [9].

Let $H = \langle V_1, \dots, V_k \rangle$ be generated by reduced words $V_1, \dots, V_k \in \mathcal{F}$. Consider a graph $\tilde{\Psi}$ which consists of k closed paths p_1, \dots, p_k such that they have a single common vertex $o = (p_i)_-$, and $|p_i| = 2|V_i|$, $i = 1, \dots, k$. Furthermore, we distinguish o as the base vertex of $\tilde{\Psi}$ and call o primary, the vertices adjacent to o are called secondary vertices and so on. The labeling function φ on p_i is defined so that $\varphi(p_i) = V_i$, $i = 1, \dots, k$, where $\varphi(p) := \varphi(e_1) \dots \varphi(e_\ell)$ if $p = e_1 \dots e_\ell$ and $e_1, \dots, e_\ell \in E\tilde{\Psi}$.

Clearly, $\tilde{\Psi} = \tilde{\Psi}_o$ is a finite connected \mathcal{A} -graph with the base vertex o that has the following property

- (Q) A word $W \in \mathcal{F}$ belongs to H if and only if there is a path p in $\tilde{\Psi}_o$ such that $p_- = p_+ = o$ and $\varphi(p) = W$.

However, $\tilde{\Psi}_o$ need not be irreducible and we will do foldings of edges in $\tilde{\Psi}_o$ which preserve property (Q) and which aim to achieve properties (P1)–(P2).

Assume that property (P1) fails for edges e, f with $e_- = f_- \in V_P\tilde{\Psi}_o$ so that $e_+ \neq f_+$ and $\theta(e) = \theta(f)$. Let us redefine the labels of all edges e' with $e'_+ = e_+$ so that $\varphi(e')\varphi(e)^{-1}$ does not change and $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$. Now we identify the edges e, f and vertices e_+, f_+ . Observe that this folding preserves property (Q) ((P2) might fail) and decreases the edge number $|E\tilde{\Psi}_o|$.

If property (P2) fails for edges e, f and $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$, then we identify the edges e, f . Note property (Q) still holds ((P1) might fail) and the number $|E\tilde{\Psi}_o|$ decreases.

Suppose property (P3) fails and there are two distinct edges e, f in $\tilde{\Psi}_o$ such that $e_- = f_-$, $e_+ = f_+ \in V_S\tilde{\Psi}_o$. By property (Q), it follows from H being factor-free that $\varphi(e) = \varphi(f)$ in $G_{\theta(e)}$. Therefore, we can identify the edges e, f , thus preserving property (Q) and decreasing the number $|E\tilde{\Psi}_o|$. If property (P3) fails so that there is a vertex v of degree 1, different from o , then we delete v along with the incident edge. Clearly, property (Q) still holds and the number $|E\tilde{\Psi}_o|$ decreases.

Thus, by induction on $|E\tilde{\Psi}_o|$, in polynomial time of size of input, which is the total length $\sum_{i=1}^k |V_i|$, we can effectively construct an irreducible \mathcal{A} -graph Ψ_o with property (Q). Other stated properties of Ψ_o are straightforward. \square

The following lemma further elaborates on the correspondence between finitely generated factor-free subgroups of the free product \mathcal{F} and irreducible \mathcal{A} -graphs.

Lemma 2. *Let Ψ_o be a finite connected irreducible \mathcal{A} -graph with the base vertex o , and $H = H(\Psi_o)$ be a subgroup of \mathcal{F} that consists of all words $\varphi(p)$, where p*

is a path in Ψ_o with $p_- = p_+ = o$. Then H is a factor-free subgroup of \mathcal{F} and $\bar{r}(H) = -\chi(\Psi_o)$, where $\chi(\Psi_o) = |V\Psi_o| - \frac{1}{2}|E\Psi_o|$ is the Euler characteristic of Ψ_o .

Proof. This follows from the facts that the fundamental group $\pi_1(\Psi_o, o)$ of Ψ_o at o is free of rank $-\chi(\Psi_o) + 1$ and that the homomorphism $\pi_1(\Psi_o, o) \rightarrow \mathcal{F}$, given by $p \rightarrow \varphi(p)$, where p is a path with $p_- = p_+ = o$, has the trivial kernel following from properties (P1)–(P2). \square

Suppose H is a nontrivial finitely generated factor-free subgroup of a free product $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$, and $\Psi_o = \Psi_o(H)$ is an irreducible \mathcal{A} -graph for H as in Lemma 1. Let $\Psi(H)$ denote the core of $\Psi_o(H)$. Clearly, $\Psi(H)$ has no vertices of degree ≤ 1 and $\Psi(H)$ is also an irreducible \mathcal{A} -graph. It is easy to see that the graph $\Psi_o(H)$ of H can be obtained back from the core graph $\Psi(H)$ of H by attaching a suitable path p to $\Psi(H)$ so that p starts at a primary vertex o , ends in $p_- \in V_P\Psi(H)$, and then by doing foldings of edges as in the proof of Lemma 1.

Now suppose H_1, H_2 are nontrivial finitely generated factor-free subgroups of \mathcal{F} . Consider a set $S(H_1, H_2)$ of representatives of those double cosets $H_1 t H_2$ of \mathcal{F} , $t \in \mathcal{F}$, that have the property $H_1 \cap t H_2 t^{-1} \neq \{1\}$. For every $s \in S(H_1, H_2)$, define the subgroup $K_s := H_1 \cap s H_2 s^{-1}$. Analogously to the case of free groups, see [17], [18], we now construct a finite irreducible \mathcal{A} -graph $\Psi(H_1, H_2)$ whose connected components are core graphs $\Psi(K_s)$, $s \in S(H_1, H_2)$.

First we define an \mathcal{A} -graph $\Psi'_o(H_1, H_2)$. The set of primary vertices of $\Psi'_o(H_1, H_2)$ is $V_P\Psi'_o(H_1, H_2) := V_P\Psi_{o_1}(H_1) \times V_P\Psi_{o_2}(H_2)$. Let

$$\tau_i : V_P\Psi'_o(H_1, H_2) \rightarrow V_P\Psi_{o_i}(H_i)$$

denote the projection map, $\tau_i((v_1, v_2)) = v_i$, $i = 1, 2$.

The set of secondary vertices $V_S\Psi'_o(H_1, H_2)$ of $\Psi'_o(H_1, H_2)$ consists of equivalence classes $[u]_\alpha$, where $u \in V_P\Psi'_o(H_1, H_2)$, $\alpha \in I$, with respect to the minimal equivalence relation generated by the following relation $\overset{\sim}{\sim}$ on $V_P\Psi'_o(H_1, H_2)$: Define $v \overset{\sim}{\sim} w$ if there are edges $e_i, f_i \in E\Psi_{o_i}(H_i)$ such that $(e_i)_- = \tau_i(v)$, $(f_i)_- = \tau_i(w)$, $(e_i)_+ = (f_i)_+$, $i = 1, 2$, the edges e_i, f_i have type α , and $\varphi(e_1)\varphi(f_1)^{-1} = \varphi(e_2)\varphi(f_2)^{-1}$ in G_α . It is easy to see that $\overset{\sim}{\sim}$ is symmetric and transitive on distinct pairs and triples (but it could lack reflexive property).

The edges in $\Psi'_o(H_1, H_2)$ are defined so that $u \in V_P\Psi'_o(H_1, H_2)$ and $[v]_\alpha \in V_S\Psi'_o(H_1, H_2)$ are connected by an edge if and only if $u \in [v]_\alpha$.

The type of a vertex $[v]_\alpha \in V_S\Psi'_o(H_1, H_2)$ is α and if $e \in E\Psi'_o(H_1, H_2)$, $e_- = u$, $e_+ = [v]_\alpha$, then $\varphi(e) := \varphi(e_1)$, where $e_1 \in E\Psi_{o_1}(H_1)$ is an edge of type α with $(e_1)_- = \tau_1(u)$, when such an e_1 exists, and $\varphi(e_1) := b_\alpha \neq 1$, $b_\alpha \in G_\alpha$, otherwise.

It follows from the definitions and properties (P1)–(P2) of $\Psi_{o_i}(H_i)$, $i = 1, 2$, that $\Psi'_o(H_1, H_2)$ is an \mathcal{A} -graph with properties (P1)–(P2). Hence, taking the core of $\Psi'_o(H_1, H_2)$, we obtain an irreducible \mathcal{A} -graph which we denote $\Psi(H_1, H_2)$.

In addition, it is not difficult to see that, when taking the connected component $\Psi'_o(H_1, H_2, o)$ of $\Psi'_o(H_1, H_2)$ that contains the vertex $o = (o_1, o_2)$ and inductively removing from $\Psi'_o(H_1 \cap H_2, o)$ vertices of degree 1 different from o , we obtain an irreducible \mathcal{A} -graph $\Psi_o(H_1 \cap H_2)$ with the base vertex o that corresponds to the intersection $H_1 \cap H_2$ as in Lemma 1.

Observe that it follows from the definitions and property (P1) for $\Psi(H_i)$, $i = 1, 2$, that, for every edge $e \in E\Psi(H_1, H_2)$ with $e_- \in V_P\Psi(H_1, H_2)$, there are unique edges $e_i \in E\Psi(H_i)$ such that $\tau_i(e_-) = (e_i)_-$, $i = 1, 2$. Hence, by setting $\tau_i(e) = e_i$,

$\tau_i(e_+) = (e_i)_+$, $i = 1, 2$, we extend τ_i to the graph map

$$\tau_i : \Psi(H_1, H_2) \rightarrow \Psi(H_i), \quad i = 1, 2.$$

It follows from definitions that τ_i is locally injective and τ_i preserves syllables of $\varphi(p)$ for every path p with primary vertices p_-, p_+ .

Lemma 3. *Suppose H_1, H_2 are finitely generated factor-free subgroups of the free product \mathcal{F} and the set $S(H_1, H_2)$ is not empty. Then the connected components of the graph $\Psi(H_1, H_2)$ are core graphs $\Psi(H_1 \cap sH_2s^{-1})$ of subgroups $H_1 \cap sH_2s^{-1}$, $s \in S(H_1, H_2)$. In particular,*

$$\bar{r}(H_1, H_2) = \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap sH_2s^{-1}) = -\chi(\Psi(H_1, H_2)).$$

Proof. As in Lemma 1, let $\Psi_{o_i}(H_i)$ be an irreducible \mathcal{A} -graph, corresponding to the subgroup H_i of \mathcal{F} , $i = 1, 2$, and $\Psi(H_i)$ denote the core of $\Psi_{o_i}(H_i)$. Let $v_i \in V_P\Psi(H_i)$, $i = 1, 2$, and $q(v_i)$ denote a path in $\Psi_{o_i}(H_i)$ with $q(v_i)_- = o_i$ and $q(v_i)_+ = v_i$. Suppose $X_i \in \mathcal{F}$, $i = 1, 2$, and $H_1^{X_1} \cap H_2^{X_2} \neq \{1\}$, where $H_i^{X_i} := X_i H_i X_i^{-1}$. Consider an irreducible \mathcal{A} -graph $\Psi_{u_i}(H_i^{X_i})$, $i = 1, 2$. Note that the core graph $\Psi(H_1^{X_1} \cap H_2^{X_2})$ can be identified with a connected component, denoted $\Psi_{(X_1, X_2)}(H_1, H_2)$, of the irreducible \mathcal{A} -graph $\Psi(H_1, H_2)$. In addition, if $w \in V_P\Psi_{(X_1, X_2)}(H_1, H_2)$, then there are paths $p_i(w)$ in $\Psi_{u_i}(H_i^{X_i})$, $i = 1, 2$, such that $(p_i(w))_- = u_i$, $(p_i(w))_+ = \tau_i(w)$, and $\varphi(p_1(w)) = \varphi(p_2(w))$. Furthermore, it follows from the definitions that $X_i \varphi(q(\tau_i(w))) \varphi(p_i(w))^{-1} \in H_i^{X_i}$, $i = 1, 2$. Therefore, there are words $V_i \in H_i$, $i = 1, 2$, such that $X_i V_i \varphi(q(\tau_i(w))) = \varphi(p_i(w))$. Since $\varphi(p_1(w)) = \varphi(p_2(w))$, we further obtain

$$X_1^{-1} X_2 = V_1 \varphi(q(\tau_1(w))) \varphi(q(\tau_2(w)))^{-1} V_2^{-1}. \quad (2)$$

Now we can draw the following conclusion. For every pair $(X_1, X_2) \in \mathcal{F} \times \mathcal{F}$ such that $H_1^{X_1} \cap H_2^{X_2} \neq \{1\}$ and a vertex $w \in V_P\Psi_{(X_1, X_2)}(H_1, H_2)$, there are words $V_i \in H_i$, $i = 1, 2$, such that the equality (2) holds. Since the paths $q(\tau_i(w))$, $i = 1, 2$, in (2) depend only on a connected component of $\Psi(H_1, H_2)$, it follows from (2) that if $(1, X), (1, Y)$ are some pairs such that $\Psi_{(1, X)}(H_1, H_2) = \Psi_{(1, Y)}(H_1, H_2)$, then $X \in H_1 Y H_2 \subseteq \mathcal{F}$.

Conversely, if $X \in H_1 Y H_2$, then the equality $\Psi_{(1, X)}(H_1, H_2) = \Psi_{(1, Y)}(H_1, H_2)$ is obviously true. Thus, the set $S(H_1, H_2)$ is in bijective correspondence with connected components of $\Psi(H_1, H_2)$ and, by Lemma 2, we have $\bar{r}(H_1 \cap sH_2s^{-1}) = -\chi(\Psi_{(1, s)}(H_1, H_2))$ for every $s \in S(H_1, H_2)$. Adding up over all $s \in S(H_1, H_2)$, we arrive to the required equality $\bar{r}(H_1, H_2) = -\chi(\Psi(H_1, H_2))$. \square

3. STRONGLY POSITIVE WORDS IN FREE PRODUCTS OF LEFT ORDERED GROUPS

Recall that G is called a *left ordered* group if G is equipped with a total order \leq which is left invariant, i.e., for every triple $a, b, c \in G$, the relation $a \leq b$ implies $ca \leq cb$. If G is left ordered, then G can also be right ordered (and vice versa). Indeed, if \leq is a left order on G then, setting $a \preceq b$ if and only if $a^{-1} \leq b^{-1}$, we obtain a right order \preceq on G .

Let G_α , $\alpha \in I$, be nontrivial left (or right) ordered groups and let $\mathcal{F} = \prod_{\alpha \in I}^* G_\alpha$ be their free product. Since it will be more convenient to work with left order, we assume that G_α , $\alpha \in I$, are left ordered. It is well known, see [12], and fairly easy

to show that there exists a total order \preceq on \mathcal{F} which extends the left orders on groups G_α , $\alpha \in I$, and which turns \mathcal{F} into a left ordered group.

A reduced word $W \in \mathcal{F}$ is called *positive* if $W \succ 1$. A reduced word W is called *strongly positive*, denoted $W \succcurlyeq 1$, if every nonempty suffix of W is positive, i.e., if $W \equiv W_1 W_2$ with $|W_2| > 0$, then $W_2 \succ 1$. Clearly, a strongly positive word is positive. Note if U, W are strongly positive and UW is reduced, then $UW \succcurlyeq 1$. A word U is (*resp. strongly*) *negative* if U^{-1} is (*resp. strongly*) positive.

Lemma 4. *Suppose S, T are strongly positive words and the word $S^{-1}T$ is reduced. Then $S^{-1}T$ is either strongly positive or strongly negative.*

Proof. Let $S \equiv S_1 S_2$, $T \equiv T_1 T_2$, where $|S_2|, |T_2| > 0$. Then $S_2 \succ 1$, $T_2 \succ 1$ by $S, T \succcurlyeq 1$. Since $S^{-1}T$ is reduced, we have $S^{-1}T \neq 1$, hence $S^{-1}T \succ 1$ or $S^{-1}T \prec 1$. Assume $S^{-1}T \succ 1$. Then $T \succ S = S_1 S_2$ or $S_1^{-1}T \succ S_2 \succ 1$, hence, in view of $T_2 \succ 1$, all nonempty suffixes of $S^{-1}T$ are positive. This implies $S^{-1}T \succcurlyeq 1$. If $S^{-1}T \prec 1$, then, switching S and T , we can show as above that $T^{-1}S \succcurlyeq 1$, hence $S^{-1}T$ is strongly negative. \square

Lemma 5. *Suppose W is a reduced word. Then there exists a factorization $W \equiv U_1 U_2^{-1}$ such that $|U_1|, |U_2| \geq 0$ and each of U_1, U_2 is either empty or strongly positive.*

Proof. Consider a factorization $W \equiv U_1^{\varepsilon_1} \dots U_k^{\varepsilon_k}$, where, for every j , $U_j \succcurlyeq 1$ and $\varepsilon_j = \pm 1$, that would be minimal with respect to k . Since for every letter a of W either $a \succcurlyeq 1$ or $a^{-1} \succcurlyeq 1$, it follows that such a factorization exists and $k \leq |W|$.

Note if $\varepsilon_j = \varepsilon_{j+1} = 1$, then $U_j U_{j+1}$ is reduced and so $U_j U_{j+1} \succcurlyeq 1$. Similarly, if $\varepsilon_j = \varepsilon_{j+1} = -1$, then $U_j^{-1} U_{j+1}^{-1}$ is reduced and so $U_{j+1} U_j \succcurlyeq 1$. Hence, it follows from the minimality of k that $\varepsilon_j \neq \varepsilon_{j+1}$ for all $j = 1, \dots, k-1$. If now $\varepsilon_j = -1$ and $\varepsilon_{j+1} = 1$ for some j , then we can use Lemma 4 and conclude that either $U_j^{-1} U_{j+1} \succcurlyeq 1$ or $U_{j+1}^{-1} U_j \succcurlyeq 1$, contrary to minimality of k . Thus, it is proven that either $k = 1$ or $k = 2$ and $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, as required. \square

Lemma 6. *Suppose W is a cyclically reduced word. Then there exists a factorization $W \equiv W_1 W_2$ such that the cyclic permutation $\bar{W} \equiv W_2 W_1$ of W is either strongly positive or strongly negative.*

Proof. By Lemma 5, $W \equiv U_1 U_2^{-1}$, where $|U_1|, |U_2| \geq 0$ and $U_j \succcurlyeq 1$ if $|U_j| > 0$, $j = 1, 2$. Since W^2 is reduced, $U_2^{-1} U_1$ is reduced and, by Lemma 4, either $U_2^{-1} U_1 \succcurlyeq 1$ or $U_2^{-1} U_1 \prec 1$. Hence, $\bar{W} \equiv U_2^{-1} U_1$ is a desired cyclic permutation of W . \square

4. PROVING THEOREM 1

Let G_α , $\alpha \in I$, be nontrivial left (or right) ordered groups and let \mathcal{F} be their free product equipped with a *left* order \preceq . Also, fix a total order \leq on the index set I .

Let Ψ be a finite irreducible \mathcal{A} -labeled graph, where $\mathcal{A} = \cup_{\alpha \in I} G_\alpha$. An edge $e \in E\Psi$ is called *maximal* if there are reduced infinite paths $p = p(e) = e_1 e_2 \dots$, $q = q(e) = f_1 f_2 \dots$ in Ψ , where $e_j, f_j \in E\Psi$, such that $e = e_1$, $(e_1)_- = (f_1)_-$ is primary, $\theta(e_1) > \theta(f_1)$, and, for every $j \geq 1$, both $\varphi(e_1 \dots e_{2j}) \prec 1$ and $\varphi(f_1 \dots f_{2j}) \prec 1$. Note that the vertices $(e_1 \dots e_{2j})_+$, $(f_1 \dots f_{2j})_+$ are primary and $q^{-1}p = \dots f_2^{-1} f_1^{-1} e_1 e_2 \dots$ is a reduced biinfinite path.

Lemma 7. *Suppose Ψ is a finite irreducible \mathcal{A} -labeled graph whose Euler characteristic is negative, $\chi(\Psi) < 0$. Then Ψ contains a maximal edge.*

Proof. Since $\chi(\Psi) < 0$, Ψ has a connected component Ψ_1 with $\chi(\Psi_1) < 0$. Without loss of generality, we may assume that $\text{core}(\Psi_1) = \Psi_1$. It is not difficult to see from $\chi(\Psi_1) < 0$ and from $\text{core}(\Psi_1) = \Psi_1$ that, for every pair $h, h' \in E\Psi_1$, there is a reduced path $p = h \dots h'$ whose first, last edges are h, h' , resp.. Pick a primary vertex o in Ψ_1 and two distinct edges t_1, u_1 with $(t_1)_- = (u_1)_- = o$. Let t, u be some reduced paths such that first edges of t, u are t_1, u_1 , resp., and t_+, u_+ have degree > 2 . Then it follows from the above remark that there are closed paths r_0, s_0 starting at t_+, u_+ , resp., such that the path $tr_0^2 t^{-1} u s_0^2 u^{-1}$ is reduced. Since Ψ_1 is irreducible and r_0, s_0 are reduced, it follows $\varphi(r_0) \neq 1, \varphi(s_0) \neq 1$ in \mathcal{F} .

Let r, s be some cyclic permutations of the closed paths r_0, s_0 , resp., that start at some primary vertices and $R = \varphi(r), S = \varphi(s)$ be reduced words. Clearly, R, S are cyclically reduced and $|R| = |r|/2 > 1, |S| = |s|/2 > 1$. By Lemma 6, there are cyclic permutations \bar{R}, \bar{S} of R, S , resp., such that $\bar{R}^{\varepsilon_r}, \bar{S}^{\varepsilon_s}$ are strongly positive, where $\varepsilon_r, \varepsilon_s \in \{\pm 1\}$. Switching from r_0, s_0 to r_0^{-1}, s_0^{-1} , resp., if necessary, we may assume that $\varepsilon_r = \varepsilon_s = -1$, i.e., $\bar{R}^{-1}, \bar{S}^{-1} \succcurlyeq 1$. Let \bar{r}, \bar{s} denote cyclic permutations of r, s , resp., such that $\varphi(\bar{r}) = \bar{R}, \varphi(\bar{s}) = \bar{S}$. Also, let $\bar{r} = \bar{r}_1 \bar{r}_2, \bar{s} = \bar{s}_1 \bar{s}_2$ be factorizations of \bar{r}, \bar{s} , resp., defined by vertices t_+, u_+ , resp..

Consider two infinite paths starting at $o = t_- = u_-$ and defined as follows. Let $T = tr_0^{+\infty}$ whose prefixes are $tr_0^k, k \geq 0$, and $U = us_0^{+\infty}$ whose prefixes are $us_0^\ell, \ell \geq 0$. It follows from the definitions that T starts at $t_- = o$, goes along t to t_+ and then cycles around r_0 infinitely many times, in particular, T is reduced. Similarly, U starts at $u_- = o$, goes along u to u_+ and then cycles around s_0 .

Denote $T = t_1 t_2 \dots$, where $t_j \in E\Psi_1$, and $U = u_1 u_2 \dots$, where $u_k \in E\Psi_1$. Let $T(j_1, j_2) := t_{j_1} \dots t_{j_2}$, where $j_1 \leq j_2$, denote the subpath of T that starts at $(t_{j_1})_-$ and ends in $(t_{j_2})_+$. It is convenient to set $T(j, j-1) := \{(t_j)_-\}$ for all $j \geq 1$. Similarly, $U(j_1, j_2) := u_{j_1} \dots u_{j_2}$, where $j_1 \leq j_2$, and $U(j, j-1) := \{(u_j)_-\}$ if $j \geq 1$.

Suppose $2j > |t| + |\bar{r}_2|$. Then $2j - |t| - |\bar{r}_2| > 0$. Let m be the remainder of $2j - |t| - |\bar{r}_2|$ when divided by $|r|$. Set $m_r := m$ if $m > 0$ and $m_r := |r|$ if $m = 0$. Note $T(1, 2j) = T(1, 2j - m_r)T(2j - m_r + 1, 2j)$ and $\varphi(T(2j - m_r + 1, 2j)) \equiv \bar{R}_3$, where \bar{R}_3 is a prefix of $\varphi(\bar{r}) \equiv \bar{R}_3 \bar{R}_4$ of even length $m_r > 0$. Recall $\varphi(\bar{r}) = \bar{R}$ and $\bar{R}^{-1} \succcurlyeq 1$, hence $\bar{R}_3^{-1} \succcurlyeq 1$ and $\bar{R}_3 \prec 1$. Note $\bar{R}_3 = \varphi(T(2j - m_r + 1, 2j)) \prec 1$ implies, by left invariance of the order \preceq , that

$$\varphi(T(1, 2j)) \prec \varphi(T(1, 2j - m_r)). \quad (3)$$

Now suppose $2j > |u| + |\bar{s}_2|$. Let m' be the remainder of $2j - |u| - |\bar{s}_2| > 0$ when divided by $|s|$. Set $m_s := m'$ if $m' > 0$ and $m_s := |s|$ if $m' = 0$. Then we can derive from $\bar{S}^{-1} \succcurlyeq 1$, similar to (3), that

$$\varphi(U(1, 2j)) \prec \varphi(U(1, 2j - m_s)). \quad (4)$$

The comparisons (3)–(4) prove that a maximal element of the infinite set

$$\{\varphi(T(1, 2j)), \varphi(U(1, 2k)) \mid j \geq 0, k \geq 1\} \subseteq \mathcal{F} \quad (5)$$

exists and it is the maximal element of the finite set

$$\{\varphi(T(1, 2j)), \varphi(U(1, 2k)) \mid 0 \leq 2j \leq |t| + |\bar{r}_2|, 0 < 2k \leq |u| + |\bar{s}_2| \} \subset \mathcal{F}.$$

Let $\varphi(Q(1, 2j_M))$, where $j_M \geq 0, Q \in \{T, U\}$, denote the maximal element of the set (5). Note $v_M = Q(1, 2j_M)_+$ is primary. Observe that elements $\varphi(T(1, 2j)), \varphi(U(1, 2k)), j \geq 0, k \geq 1$, in (5) are distinct and represent φ -labels of subpaths of the biinfinite path $U^{-1}T$ that connect the primary vertex $o = t_-$ to all primary vertices of $U^{-1}T$ along $U^{-1}T$ (or its inverse). If we take another primary vertex v on

$U^{-1}T$ and consider the set of labels of subpaths that connect v to primary vertices of $U^{-1}T$ as above, then the resulting set can be obtained from (5) by multiplication on the left by $\varphi(h(o, v))^{-1}$, where $h(o, v) = T(1, 2j_v)$ if $v = T(1, 2j_v)_+$, $2j_v \geq 0$, and $h(o, v) = U(1, 2k_v)$ if $v = U(1, 2k_v)_+$, $2k_v > 0$. Since left multiplication preserves the order, these remarks imply that the vertex $v_M = Q(1, 2j_M)_+$ defines a factorization of the biinfinite path $U^{-1}T = q^{-1}p$ into infinite paths q, p , where $p = e_1e_2 \dots$, $q = f_1f_2 \dots$, e_j, f_j are edges, $j \geq 1$, so that, for every $j \geq 1$, we have $\varphi(e_1e_2 \dots e_{2j}) \prec 1$ and $\varphi(f_1f_2 \dots f_{2j}) \prec 1$. Therefore, if $\theta(e_1) > \theta(f_1)$, then e_1 is a maximal edge of Ψ and if $\theta(f_1) > \theta(e_1)$, then f_1 is maximal in Ψ . \square

Suppose Γ is a finite graph. A set $D \subseteq E\Gamma$ of edges is called *good* (for cutting) if the graph $\Gamma \setminus (D \cup D^{-1})$ consists of connected components whose Euler characteristics are 0. Clearly, Γ contains a good edge set if and only if no connected component of Γ is a tree.

Lemma 8. *Suppose Ψ is a finite connected irreducible \mathcal{A} -graph with $\chi(\Psi) < 0$. Then the set of all maximal edges of Ψ is good (for cutting).*

Proof. Arguing on the contrary, assume that $D \subseteq E\Psi$ is the set of all maximal edges of Ψ and D is not good. Then the graph $\Psi \setminus (D \cup D^{-1})$ contains a connected component Ψ'_1 with either $\chi(\Psi'_1) < 0$ or $\chi(\Psi'_1) > 0$. If $\chi(\Psi'_1) < 0$, then the core $\Psi_1 = \text{core}(\Psi'_1)$ of Ψ'_1 is a finite irreducible \mathcal{A} -graph with $\chi(\Psi_1) < 0$. By Lemma 7, Ψ_1 contains a maximal edge e . However, it follows from the definition that e is also maximal for Ψ , hence, $e \in D$. This contradiction shows that $\chi(\Psi'_1) > 0$, hence Ψ'_1 is a tree which we denote T .

Let C denote the set that consists of all $c \in E\Psi$ such that $c_+ \in VT$ and $c \notin ET$. It follows from the definitions that if $c \in C$ then $c \in D$ or $c^{-1} \in D$. Since every $d \in D$ is maximal, there are infinite reduced paths $p(d) = e_1(d)e_2(d) \dots$ and $q(d) = f_1(d)f_2(d) \dots$ such that $e_1(d)_- = f_1(d)_- \in V_P\Psi$, $e_1(d) = d$, $\theta(e_1(d)) > \theta(f_1(d))$ and, for every $j \geq 1$, $\varphi(e_1(d) \dots e_{2j}(d)) \prec 1$ and $\varphi(f_1(d) \dots f_{2j}(d)) \prec 1$.

Pick an arbitrary $c \in C$. Suppose c_- is primary. Since $d_- \in V_P\Psi$ if $d \in D$ and c or c^{-1} is in D , it follows that $c \in D$. Consider a shortest path of the form $h(c) := e_1(c) \dots e_{2\ell}(c)$, $\ell \geq 1$, such that either $e_{2\ell}(c)^{-1} \in C$ or $e_{2\ell}(c)^{-1} \in T$ and $e_{2\ell+1}(c)^{-1} \in C$. Define $\sigma(c) := e_{2\ell}(c)^{-1}$ if $e_{2\ell}(c)^{-1} \in C$ and $\sigma(c) := e_{2\ell+1}(c)^{-1}$ if $e_{2\ell+1}(c)^{-1} \in C$. Since T is a finite tree, such a path $h(c)$ exists, $|h(c)| > 0$, $\sigma(c) \neq c$, and $\varphi(h(c)) \prec 1$. Note $h(c)_-$, $h(c)_+$ are primary vertices of c , $\sigma(c)$, resp., and $h(c) = ch_T(c)\sigma(c)^{-\varepsilon_{\sigma(c)}}$, where $h_T(c)$ is a subpath of $h(c)$ in T with $h_T(c)_- = c_+$, $h_T(c)_+ = \sigma(c)_+$, and $\varepsilon_{\sigma(c)} = 1$ if $\sigma(c)_+$ is secondary and $\varepsilon_{\sigma(c)} = 0$ if $\sigma(c)_+$ is primary.

Now assume that c_+ is primary. Then $c^{-1} = d_c \in D$. Consider a shortest path of the form $h(c) := f_1(d_c) \dots f_{2\ell-2}(d_c)$, where $\ell \geq 1$ and if $\ell = 1$ then $h(c) := \{c_+\}$, such that either $f_{2\ell-2}(d_c)^{-1} \in C$ or $f_{2\ell-2}(d_c)^{-1} \in T$ (or $f_{2\ell-2}(d_c)$ is undefined if $\ell = 1$) and $f_{2\ell-1}(d_c)^{-1} \in C$. Define $\sigma(c) := f_{2\ell-2}(d_c)^{-1}$ if $f_{2\ell-2}(d_c)^{-1} \in C$ and $\sigma(c) := f_{2\ell-1}(d_c)^{-1}$ if $f_{2\ell-1}(d_c)^{-1} \in C$. Since T is a finite tree, such a path $h(c)$ exists, $|h(c)| \geq 0$, $\sigma(c) \neq c$, and $\varphi(h(c)) \preceq 1$. In addition, the equality $\varphi(h(c)) = 1$ implies that $h(c) = \{c_+\}$, $\sigma(c) = f_1(c)^{-1}$ and $\theta(\sigma(c)) = \theta(f_1(d_c)) < \theta(e_1(d_c)) = \theta(c)$. As above, we remark that $h(c)_-$, $h(c)_+$ are primary vertices of c , $\sigma(c)$, resp., and $h(c) = h_T(c)\sigma(c)^{-\varepsilon_{\sigma(c)}}$, where $h_T(c)$ is in T with $h_T(c)_- = c_+$, $h_T(c)_+ = \sigma(c)_+$, and $\varepsilon_{\sigma(c)} = 1$ if $\sigma(c)_+$ is secondary and $\varepsilon_{\sigma(c)} = 0$ if $\sigma(c)_+$ is primary.

Let us summarize. For every $c \in C$, we have defined an edge $\sigma(c) \in C$, $\sigma(c) \neq c$, hence, $\sigma : C \rightarrow C$ is a function. Furthermore, there is a reduced path $h(c)$ such that $h(c) = c^{\varepsilon_c} h_T(c) \sigma(c)^{-\varepsilon_{\sigma(c)}}$, where $h_T(c)$ is in T with $h_T(c)_- = c_+$, $h_T(c)_+ = \sigma(c)_+$, $\varepsilon_c = 1$ if $c_+ \in V_S \Psi$ and $\varepsilon_c = 0$ if $c_+ \in V_P \Psi$. In addition, $\varepsilon_{\sigma(c)} = 1$ if $\sigma(c)_+ \in V_S \Psi$ and $\varepsilon_{\sigma(c)} = 0$ if $\sigma(c)_+ \in V_P \Psi$. Also, $\varphi(h(c)) \preceq 1$ and $\varphi(h(c)) = 1$ implies that $h(c) = \{c_+\} = \{\sigma(c)_+\}$ and $\theta(c) > \theta(\sigma(c))$. Finally, $h(c)_-$, $h(c)_+$ are primary vertices of c , $\sigma(c)$, resp., whence $h(c)_+ = h(\sigma(c))_-$ for every $c \in C$.

Since C is finite, there is a cycle $c, \sigma(c), \dots, \sigma^k(c) = c$, $k \geq 2$, for some $c \in C$. Consider the closed path $p_c = h(c)h(\sigma(c)) \dots h(\sigma^{k-1}(c))$. Since $\varphi(h(\sigma^j(c))) \preceq 1$ for every j , we obtain that $\varphi(p_c) \preceq 1$ and the equality $\varphi(p_c) = 1$ implies $\varphi(h(\sigma^j(c))) = 1$ and $h(\sigma^j(c)) = \{\sigma^j(c)_+\}$ for every j . On the other hand,

$$p_c = c^{\varepsilon_c} h_T(c) \sigma(c)^{-\varepsilon_{\sigma(c)}} \sigma(c)^{\varepsilon_{\sigma(c)}} h_T(\sigma(c)) \sigma^2(c)^{-\varepsilon_{\sigma^2(c)}} \dots \\ \sigma^{k-1}(c)^{\varepsilon_{\sigma^{k-1}(c)}} h_T(\sigma^{k-1}(c)) \sigma^k(c)^{-\varepsilon_{\sigma^k(c)}} = c^{\varepsilon_c} h_T(c) h_T(\sigma(c)) \dots h_T(\sigma^{k-1}(c)) c^{-\varepsilon_c}$$

following from $\sigma^k(c) = c$. Since $h_T(c)h_T(\sigma(c)) \dots h_T(\sigma^{k-1}(c))$ is a closed path in the tree T , we have $\varphi(p_c) = 1$ in \mathcal{F} . Therefore, $h(\sigma^j(c)) = \{\sigma^j(c)_+\} = \{\sigma(c)_+\}$ and $\theta(\sigma^j(c)) > \theta(\sigma^{j+1}(c))$ for every $j = 0, 1, \dots, k-1$, implying $\theta(c) > \theta(\sigma^k(c)) = \theta(c)$. This contradiction completes the proof. \square

Proof of Theorem 1. As in Sect. 2, consider a finite irreducible \mathcal{A} -graph $\Psi(H_1, H_2)$ whose connected components correspond to core graphs of subgroups $H_1 \cap sH_2s^{-1}$, $s \in S(H_1, H_2)$. Without loss of generality, we may assume that $-\chi(\Psi(H_1, H_2)) > 0$. Let D be the set of all maximal edges in $\Psi(H_1, H_2)$. It is easy to see from the definitions that if $d \in D$, then $\tau_i(d)$ is maximal in $\Psi_{o_i}(H_i)$, $i = 1, 2$. Hence, $\tau_i(D) \subseteq D_i$, where D_i is the set of maximal edges of $\Psi_{o_i}(H_i)$, $i = 1, 2$. By Lemma 8, D_i is good for $\Psi_{o_i}(H_i)$ and it follows from Lemma 3 and definitions that

$$\bar{r}(H_1, H_2) = |D| \leq |\tau_1(D)| \cdot |\tau_2(D)| \leq |D_1| \cdot |D_2| = \bar{r}(H_1) \cdot \bar{r}(H_2),$$

as desired. \square

5. THE FREE GROUP CASE

Suppose H_1, H_2 are finitely generated subgroups of a free group $F = F(\mathcal{A})$, where $\mathcal{A} = \{a_1, \dots, a_m\}$ is a set of free generators of F . Let $F(a, b)$ be a free group of rank 2 with free generators a, b . Note that the map $\mu : a_i \rightarrow a^i b^i a^{-i} b^{-i}$, $i = 1, \dots, m$, extends to a monomorphism $\mu : F(\mathcal{A}) \rightarrow F(a, b)$ such that $\mu(H_1), \mu(H_2)$ are factor-free subgroups of the free product $F(a, b) = A * B$, where $A = \langle a \rangle$, $B = \langle b \rangle$ are infinite cyclic groups generated by a, b . We may assume that $\mu(S(H_1, H_2)) \subseteq S(\mu(H_1), \mu(H_2))$. Since a cyclic group is left ordered, it follows from Theorem 1 that

$$\bar{r}(H_1, H_2) = \sum_{s \in S(H_1, H_2)} \bar{r}(H_1 \cap sH_2s^{-1}) \leq \sum_{t \in S(\mu(H_1), \mu(H_2))} \bar{r}(\mu(H_1) \cap t\mu(H_2)t^{-1}) \\ \leq \bar{r}(\mu(H_1))\bar{r}(\mu(H_2)) = \bar{r}(H_1)\bar{r}(H_2). \quad (6)$$

We remark that there is a more direct way to prove the inequality (6) by repeating verbatim the arguments of Sect. 4 with a few changes in basic definitions. To do this, consider a graph U with $VU = \{o_P, o_S\}$ and $EU = \{a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}\}$, where $(a_j)_- = o_P$ and $(a_j)_+ = o_S$, $j = 1, 2, 3$. The fundamental group $F_2 = \pi_1(U, o_P)$ of U at o_P is free of rank 2, and $F_2 = \langle a_1 a_2^{-1}, a_1 a_3^{-1} \rangle$ is freely generated by $a_1 a_2^{-1}, a_1 a_3^{-1}$. Let H_1, H_2 be finitely generated subgroups of F_2 , X_i be the Stallings graph of

H_i , $i = 1, 2$, and W be the core of the pullback $X_1 \times_U X_2$ of X_1, X_2 over U , see [18]. If $Q \in \{X_1, X_2, W, U\}$, there is a canonical graph map $\varphi_Q : Q \rightarrow U$ which is locally injective and which we call *labeling*. If $v \in VQ$ and $\varphi_Q(v) = o_P$, v is called *primary*. If $\varphi_Q(v) = o_S$, v is *secondary*. The image $\varphi_Q(e) = a_j^{\pm 1}$ is the *label* of an edge $e \in EQ$ and $\theta(e) := j \in \{1, 2, 3\} = I$ is the *type* of e . With this terminology, the definitions and arguments of Sect. 4 for graphs Q, W, X_1, X_2 and the group F_2 in place of $\Psi, \Psi(H_1, H_2), \Psi_{o_1}(H_1), \Psi_{o_2}(H_2)$ and \mathcal{F} , resp., are retained.

REFERENCES

- [1] Y. Antolín, A. Martino, and I. Schwabrow, *Kurosh rank of intersections of subgroups of free products of right-orderable groups*, preprint, <http://arxiv.org/abs/1109.0233v3>
- [2] W. Dicks, *Equivalence of the strengthened Hanna Neumann conjecture and the amalgamated graph conjecture*, Invent. Math. **117**(1994), 373–389.
- [3] W. Dicks, *Simplified Mineyev*, preprint, <http://mat.uab.cat/~dicks/SimplifiedMineyev.pdf>
- [4] W. Dicks and S. V. Ivanov, *On the intersection of free subgroups in free products of groups*, Math. Proc. Cambridge Phil. Soc. **144**(2008), 511–534.
- [5] W. Dicks and S. V. Ivanov, *On the intersection of free subgroups in free products of groups with no 2-torsion*, Illinois J. Math. **54**(2010), 223–248.
- [6] J. Friedman, *Sheaves on graphs, their homological invariants, and a proof of the Hanna Neumann conjecture: with an appendix by Warren Dicks*, Mem. Amer. Math. Soc. **233**(2014), no. 1100. xii+106 pp.
- [7] S. V. Ivanov, *On the intersection of finitely generated subgroups in free products of groups*, Internat. J. Algebra and Comp. **9**(1999), 521–528.
- [8] S. V. Ivanov, *Intersecting free subgroups in free products of groups*, Internat. J. Algebra and Comp. **11**(2001), 281–290.
- [9] S. V. Ivanov, *On the Kurosh rank of the intersection of subgroups in free products of groups*, Adv. Math. **218**(2008), 465–484.
- [10] S. V. Ivanov, *A property of groups and the Cauchy-Davenport theorem*, J. Group Theory **13**(2010), 21–39.
- [11] S. V. Ivanov, *Linear programming and the intersection of free subgroups in free products of groups*, submitted.
- [12] V. M. Kopytov and N. Y. Medvedev, *Right ordered groups*, Plenum Publ., New York, 1996.
- [13] A. G. Kurosh, *The theory of groups*, Chelsea, 1956.
- [14] R. C. Lyndon and P. E. Schupp, *Combinatorial group theory*, Springer-Verlag, 1977.
- [15] I. Mineyev, *Submultiplicativity and the Hanna Neumann conjecture*, Ann. Math. **175**(2012), 393–414.
- [16] H. Neumann, *On the intersection of finitely generated free groups*, Publ. Math. **4**(1956), 186–189; Addendum, Publ. Math. **5**(1957), 128.
- [17] W. D. Neumann, *On the intersection of finitely generated subgroups of free groups*, Lecture Notes in Math. (Groups-Canberra 1989) **1456**(1990), 161–170.
- [18] J. R. Stallings, *Topology of finite graphs*, Invent. Math. **71**(1983), 551–565.

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